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Topology and its Applications

www.elsevier.com/locate/topolCombinatorial modulus and type of graphs[☆]

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ARTICLE INFO

Article history:

Received 6 October 2006

Received in revised form 2 February 2009

Accepted 2 February 2009

Keywords:

Discrete conformal geometry

Extremal length

Conformal modulus

Type problem

Outer spheres

ABSTRACT

Let A be the 1-skeleton of a triangulated topological annulus. We establish bounds on the combinatorial modulus of a refinement A' , formed by attaching new vertices and edges to A , that depend only on the refinement and not on the structure of A itself. This immediately applies to showing that a disk triangulation graph may be refined without changing its combinatorial type, provided the refinement is not too wild. We also explore the type problem in terms of disk growth, proving a parabolicity condition based on a superlinear growth rate, which we also prove optimal. We prove our results with no degree restrictions in both the EEL and VEL settings and examine type problems for more general complexes and dual graphs.

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1. Introduction

There are two ways to carry the notion of conformal modulus of a ring domain to a triangulated annulus, depending on whether metrics are assigned to the vertices or the edges of the 1-skeleton. The two versions are qualitatively different and even lead to inequivalent notions of discrete conformal type – VEL type for vertices, EEL type for edges.

The goal of this paper is to establish how subdividing the faces of a triangulated annulus can affect its discrete modulus in either setting. We show that the distortion of the modulus may be bounded in terms of the subdivision alone with no dependence on the original triangulation. In particular, there is no dependence on degree. This is avoided by applying an observation of Chrobak and Eppstein [2] that any planar graph may be considered as a directed graph with globally bounded outdegree. This weaker notion of bounded degree is sufficient to control the bounds.

This has immediate application to discrete type problems, i.e. determining whether a disk triangulation graph is hyperbolic or parabolic in either the VEL or EEL setting. We show that if a disk triangulation graph is refined in a sufficiently reasonable way, the resulting graph will have the same type. We obtain different notions of “sufficiently reasonable” for VEL and EEL types, but both will cover most standard refinement processes, such as hexagonal and barycentric subdivision.

We then turn to exploring discrete type in terms of the growth of spheres. There are already some results of this ilk (e.g., [7,10,11]), but they require symmetry or degree restrictions on the graph. We obtain a superlinear growth condition that guarantees parabolicity with no such restrictions. We also show how to construct slow-growing (e.g., $n^{1+\varepsilon}$) hyperbolic graphs, establishing sharpness of the parabolicity condition.

We establish our definitions and foundational lemmas in Section 2. Our main results regarding the moduli of ring domains are developed and proved in Section 3. We apply these results to the type problem in Section 4 and offer examples demonstrating the necessity of our hypotheses. We show how to generalize our results to non-triangular complexes in Section 5 and apply this result to relate the type of a complex to that of its dual. We also introduce discrete outer spheres and explore their application to discrete type problems. Our results relating type to sphere growth are covered in Section 6.

[☆] This paper documents results of the author's PhD thesis [14] prepared at Florida State University under the direction of Philip L. Bowers, whose guidance and support were invaluable to this work.

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2. Preliminaries

2.1. Extremal length

Our definitions for combinatorial extremal length are consistent with [6].

Let X be a non-empty set and Γ a non-empty collection of finite or infinite sequences in X , called *paths*. We will thus refer to the pair (X, Γ) . A *metric* on X is a function $m : X \rightarrow [0, \infty)$. The value $m(x)$ is the m -weight or m -measure of x . The *area* of m is

$$\text{area}(m) = \sum_{x \in X} m(x)^2,$$

and a metric is called *admissible* if it has finite, non-zero area. Let $\mathcal{M}(X) = \{m : 0 < \text{area}(m) < \infty\}$ be the set of admissible metrics on X . For a path $A = \{a_0, a_1, \dots\} \subset X$ and $m \in \mathcal{M}(X)$, define the m -length to be $L_m(A) = \sum_{j=1}^{\infty} m(a_j) = \sum_{x \in A} m(x)$. For a collection Γ of paths in X , define $L_m(\Gamma) = \inf_{A \in \Gamma} L_m(A)$ and the extremal length

$$\text{EL}(\Gamma) = \sup_{m \in \mathcal{M}(X)} \left\{ \frac{L_m(\Gamma)^2}{\text{area}(m)} \right\}.$$

The reciprocal of extremal length is the *modulus*.

We say Γ is *hyperbolic* if $\text{EL}(\Gamma)$ is finite and *parabolic* if $\text{EL}(\Gamma)$ is infinite. When the set Γ is clear from the context, we write $\text{EL}(X) = \text{EL}(\Gamma)$.

An *extremal metric* for Γ is an admissible metric μ on X for which $\text{EL}(\Gamma) = \frac{L_\mu(\Gamma)^2}{\text{area}(\mu)}$. In the case $\text{EL}(\Gamma) = \infty$, an extremal metric has finite area and all elements of Γ have infinite length.

Lemma 2.1. *Let X be set and Γ a collection of subsets. If Γ is finite or if $\text{EL}(\Gamma) = \infty$, then there is an extremal metric for Γ on X .*

The finite case is proved in [1]. The latter case defines a *parabolic extremal metric*; its existence is an exercise in [6].

Note that scaling the metric does not change the quantity maximized by extremal length and so we may assume that our metrics are always normalized to have area one.

He and Schramm also offer in [6] the important *monotonicity property*, stated as

Lemma 2.2. *Suppose Γ and Γ' are collections of subsets of X with the property that for every $\gamma \in \Gamma$ there is a $\gamma' \in \Gamma'$ such that $\gamma' \subset \gamma$. (In particular, this holds if $\Gamma \subset \Gamma'$.) Then $\text{EL}(\Gamma') \leq \text{EL}(\Gamma)$.*

2.2. Comparability

Let f and g be positive real-valued functions with common domain \mathcal{Y} . Let $k \geq 1$. We say the functions are k -comparable if $\frac{1}{k}g(v) \leq f(v) \leq kg(v)$ for all $v \in \mathcal{Y}$. f and g are *comparable* if they are k -comparable for some $k \geq 1$. It is easy to verify that comparability defines an equivalence relation, and this is the relation we seek when determining finiteness of extremal length. Its application is prescribed by the following lemma.

Lemma 2.3. *Let X and Y be infinite sets with sets of paths Γ_X and Γ_Y . Let $\{A_i\}_{i=0}^{\infty}$ and $\{B_i\}_{i=0}^{\infty}$ be collections of finite subsets of X and Y , respectively, and $\Gamma_X^i = \{\gamma \cap A_i : \gamma \in \Gamma_X\}$ and $\Gamma_Y^i = \{\gamma \cap B_i : \gamma \in \Gamma_Y\}$. Suppose these sets satisfy the following properties:*

1. $X = \bigcup_{i \geq 0} A_i$ and $Y = \bigcup_{i \geq 0} B_i$.
2. If $i < j$, then $A_i \subset A_j$ and $B_i \subset B_j$.
3. Let $i \geq 0$. For every $\gamma_X \in \Gamma_X$ and $\gamma_Y \in \Gamma_Y$, $\gamma_X \cap A_i$ and $\gamma_Y \cap B_i$ are non-empty.
4. $\text{EL}(\Gamma_X^i)$ and $\text{EL}(\Gamma_Y^i)$ are comparable (taken as functions of i).

Then $\text{EL}(\Gamma_X) = \infty$ if and only if $\text{EL}(\Gamma_Y) = \infty$.

Proof. Suppose $\text{EL}(\Gamma_X) = \infty$ with unit-area parabolic extremal metric μ . Define μ_i to be the restriction of μ to A_i and note that $\text{area}(\mu_i) \leq \text{area}(\mu) = 1$. Choose any $N > 0$. We show $\text{EL}(\Gamma_Y) > N$, implying $\text{EL}(\Gamma_Y) = \infty$.

Since X is parabolic, every element $\gamma \in \Gamma_X$ has infinite μ -length. Choose $k > 0$ so that $k \text{EL}(\Gamma_X^i) \leq \text{EL}(\Gamma_Y^i)$ for all $i > 0$. All paths in Γ_X have infinite μ -length, and so for any given path $\gamma \in \Gamma_X$ there is a $j_\gamma > 0$ so that $L_{\mu_{j_\gamma}}(\gamma \cap A_j) > \sqrt{\frac{N}{k}}$. Let $j = \inf_{\gamma \in \Gamma_X} j_\gamma$. Since every path in Γ_X^j is contained in a path in Γ_X , we have $L_{\mu_j}(\Gamma_X^j) > \sqrt{\frac{N}{k}}$. Then

$$N < k L_{\mu_j}(\Gamma_X^j)^2 \leq k \frac{L_{\mu_j}(\Gamma_X^j)^2}{\text{area}(\mu_j)} \leq k \sup_{m \in \mathcal{M}(A_j)} \frac{L_m(\Gamma_X^j)^2}{\text{area}(m)} = k \text{EL}(\Gamma_X^j) \leq \text{EL}(\Gamma_Y^j) \leq \text{EL}(\Gamma_Y).$$

The last inequality is a direct consequence of monotonicity (Lemma 2.2). The proof is completed by repeating the argument with the roles of X and Y exchanged. \square

3. Refinement and extremal length

3.1. Shadow paths

Our goal is to control the combinatorial extremal length of a set X that is related to some other set X' whose extremal length is known. We codify our technique in the following lemma.

Lemma 3.1. *Let X' be a set, Γ' a collection of subsets of X' , and μ' an extremal metric for (X', Γ') . Suppose there is a set X , a collection Γ of subsets of X , an admissible metric μ on X , and constants $C, D > 0$ with the following properties:*

1. $\text{area}(\mu) \leq C \cdot \text{area}(\mu')$.
2. For each $\gamma \in \Gamma$, there is a $\gamma' \in \Gamma'$ such that $D \cdot L_{\mu'}(\gamma') \leq L_{\mu}(\gamma)$.

Then $\text{EL}(\Gamma) \geq \frac{D^2}{C} \text{EL}(\Gamma')$.

Proof. We associate to each $\gamma \in \Gamma$ a specific path $\gamma' \in \Gamma'$ with $L_{\mu}(\gamma) \geq D \cdot L_{\mu'}(\gamma')$. Let $\Gamma^{\#}$ be the collection of these γ' . The proof now amounts to unraveling the definitions.

$$\begin{aligned} \text{EL}(\Gamma) &= \sup_{m \in \mathcal{M}(X)} \frac{\inf_{\gamma \in \Gamma} L_m(\gamma)^2}{\text{area}(m)} \geq \frac{\inf_{\gamma \in \Gamma} L_{\mu}(\gamma)^2}{\text{area}(\mu)} \\ &\geq \frac{\inf_{\gamma' \in \Gamma^{\#}} (D \cdot L_{\mu'}(\gamma'))^2}{\text{area}(\mu)} \geq D^2 \frac{\inf_{\gamma' \in \Gamma'} L_{\mu'}(\gamma')^2}{\text{area}(\mu)} \\ &\geq D^2 \frac{\inf_{\gamma' \in \Gamma'} L_{\mu'}(\gamma')^2}{C \cdot \text{area}(\mu')} = \frac{D^2}{C} \cdot \frac{\inf_{\gamma' \in \Gamma'} L_{\mu'}(\gamma')^2}{\text{area}(\mu')} = \frac{D^2}{C} \text{EL}(\Gamma'). \quad \square \end{aligned}$$

The set $\Gamma^{\#}$ is the set of *shadow paths* and is essential to the forthcoming results. Suppose we want to find the extremal length of a pair (X, Γ) that is constructed from another pair (X', Γ') whose extremal length is known. Assume an extremal metric μ' on X' . We then use μ' to construct a new metric μ on X satisfying the assumptions of Lemma 3.1, meaning we always have two things to control: area and path length. The trick is to construct the metric so that the constants C and D depend on as little as possible.

For $x \in X$, $\mu(x)$ is assigned a value of $\mu(x')$ for some $x' \in X'$. That is, each element $x \in X$ has a corresponding element $x' \in X'$ that prescribes its measure. The constant C is a bound on the number of elements in X to which an element of X' may be assigned.

For a path γ in Γ , we must guarantee a path in $\gamma' \in \Gamma'$ whose μ' -length is less than $\frac{1}{D}$ times the μ -length of γ . The paths γ and γ' naturally correspond. As γ bobs and weaves through X , γ' will “shadow” its movement in X' and have comparable length.

These two conditions are at odds. We must choose μ carefully so that paths are sufficiently long, but so that the area stays sufficiently small.

Our objective is comparability of the extremal lengths of two sets A and B . This requires applying Lemma 3.1 twice, with A and B alternatively taking the roles of X and X' . The extremal lengths of A and B are shown to be k -comparable for some k depending only on the constants in the lemma.

3.2. Graphs

A graph $G = (V, E)$ is a set V of vertices and a set E of edges. An edge connecting two vertices v_0 and v_1 is denoted $[v_0, v_1]$, and the vertices v_0 and v_1 are the *endpoints*. A graph is finite or infinite as $|V|$ is finite or infinite. We assume our graphs are connected and have no multiple edges or vertex self-loops. Two vertices v, w are *adjacent*, denoted $v \sim w$, if $[v, w] \in E$. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic*, denoted $G \cong G'$, if there is a bijection $f: V \rightarrow V'$ such that for any $v, w \in V$ we have $v \sim w$ if and only if $f(v) \sim f(w)$. Every vertex has a *degree* $\deg(v_0) = |\{v \in V: [v, v_0] \in E\}|$, the number of edges having v_0 as an endpoint. We will assume the degree is finite for each vertex (the graph is *locally finite*). The degree of a graph is defined as $\sup_{v \in V} \deg(v)$. The graph has *bounded degree* if the degree is finite, *unbounded degree* otherwise.

A *vertex path* in G is a finite or infinite sequence of vertices v_0, v_1, \dots such that for all $i \geq 0$, either $v_i \sim v_{i+1}$ or $v_i = v_{i+1}$. Similarly, an *edge path* is a sequence of edges of the form $[v_0, v_1], [v_1, v_2], [v_2, v_3], \dots$, i.e. consecutive edges laid end to end. Note that each vertex path has a corresponding edge path, and vice versa. The combinatorial vertex length and edge

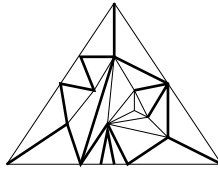


Fig. 1. Bold edges are incident-adjacent.

length of a path are the numbers of vertices and edges in the path, respectively. If a base point $v_0 \in V$ is specified, we say the *norm* $|w|$ of a vertex $w \in V$ is the combinatorial vertex length of the shortest path connecting v_0 to w . A *cycle graph* is a finite connected graph whose vertices all have degree two.

We generally think of the vertices as points and the edges as arcs connecting the endpoints. A graph is *planar* if the graph can be embedded in the plane – that is, the vertex points and edge arcs may be positioned in the plane so that the vertices are located at distinct points and the edge arcs intersect only at shared endpoints. An embedding like this is a *diagram* for a graph. For example, the diagram of a cycle is a Jordan curve.

Our interest in graphs is to mimic the geometric properties of classical Riemann surfaces with a combinatorial object. This is achieved via the *triangulation graph*, which is the 1-skeleton of a locally finite tiling by triangles of a simply connected Riemann surface (possibly with boundary). A triangle with vertices or edges a, b, c is denoted $\Delta(a, b, c)$. Of specific interest are *disk triangulation graphs*, in which the Riemann surface in question is the open unit disk. We also study more general *disk cell complexes* whose 1-skeletons are *disk cell graphs*. These are obtained from locally finite tilings of the disk by general finite polygons, not necessarily triangles. The interiors of the polygons are the *faces* or *cells*. We frequently require a global bound on the number of sides of the polygonal faces. When a cell structure is present and relevant, we will expand the graph notation $G = (V, E, F)$ to include the set of polygonal faces F .

A *directed graph* is a graph $G = (V, E)$ along with an ordering on the vertices of each edge. For an edge $[x, y] \in E$, we may form a *directed edge* by $[x, y]$, where x is the *tail vertex* and y is the *head*. For any vertex $v \in V$, the *outdegree* of v is the number of directed edges for which v is the tail vertex.

Let G be a cell complex embedded in the plane with disjoint subcomplexes $A, B, C \subset G$. We say C *separates* A from B if A and B lie in different components of $G \setminus C$ and C *separates* A from infinity if A lies in an unbounded component of $G \setminus C$.

A planar graph is a *combinatorial annulus* or an *annular complex* if the graph is the 1-skeleton of triangulated topological annulus in the plane, i.e. a region $A \subset \mathbb{C}$ whose boundary consists of two disjoint Jordan curves C_1, C_2 such that the bounded component of $\mathbb{C} \setminus C_2$ contains the bounded component of $\mathbb{C} \setminus C_1$.

We may apply the definition of combinatorial extremal length to a graph $G = (V, E)$ in two ways, according to whether the metric assigns values to the set of vertices or to the set of edges (*vertex metrics* and *edge metrics*).

Let G be a combinatorial annulus and let $\Gamma_V = \Gamma_V(G)$ be the set of vertex paths connecting the two boundary components of G . Define $\Gamma_E = \Gamma_E(G)$ similarly for edge paths. Define the *vertex extremal length* $\text{VEL}(G) = \text{EL}(\Gamma_V)$, and the *edge extremal length* $\text{EEL}(G) = \text{EL}(\Gamma_E)$. A graph is *VEL-hyperbolic* or *VEL-parabolic* as the vertex extremal length is finite or infinite, indicating its *VEL type*. *EEL-hyperbolic*, *EEL-parabolic*, and *EEL type* are defined similarly.

These definitions offer a discrete analog to the classical type problem for Riemann surfaces. The two definitions of type are equivalent for graphs of bounded degree, but it is not difficult to construct a graph that is VEL-parabolic and EEL-hyperbolic (see [6]). EEL type indicates recurrence or transience of a random walk or electric network, whereas VEL type corresponds to circle packing type (see [5,4,6,12]). We will also employ a connection to square tilings [3,9].

3.3. Refinement

Let $G = (V, E, F)$, $rG = (rV, rE, rF)$ be planar complexes and suppose there is an injection $\iota: V \rightarrow rV$ with the property that if $[x, y] \in E$, there is a collection of vertices $w_0 = \iota(x), w_1, \dots, w_n = \iota(y)$ with $[w_j, w_{j+1}] \in rE$ and $w_j \notin \iota(V)$ for each $j \neq 0, n$. We then call rG a *refinement* of G and we shall consider the refinement r as a map from an appropriate set of graphs to itself such that rG is always a refinement of G . We abuse notation by suppressing further mention of the injection ι and considering $V \subset rV$.

Less technically, a refinement attaches vertices to the edges of a graph, dividing the edge into subedges, and then adds edges inside the faces.

We require some notation before specifying the classes of refinements we need to consider. Let $G = (V, E, F)$ and r a refinement of G . For $e = [x, y] \in E$, we refer to the vertices in rV incident to e as the set $\text{vInc}_r(e) = \{w_1, \dots, w_{n-1}\}$ guaranteed by the definition of refinement (note that we do not include the endpoints), and similarly the set of edges incident to e is $\text{eInc}_r(e) = \{[w_0, w_1], \dots, [w_{n-1}, w_n]\}$. We also consider edges that are *incident-adjacent*, $\text{eIncAdj}_r(e) = \{[x', y'] \in rE: x' \in \text{vInc}_r(e) \cup \{x, y\} \text{ and } [x', y'] \text{ is contained in a cell bounded by } e\}$. This definition requires G to have a planar cell structure. See Fig. 1. For any vertex $v \in rV \setminus V$ not incident to an edge of G , there is a face F of G whose incident edges separate v from infinity. We say v then *lies in* F . Similarly, an edge lies in F if either of its endpoints lies in F and it is not incident to any edge of G .

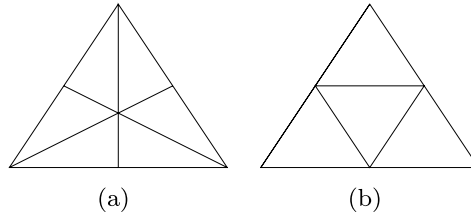


Fig. 2. (a) Barycentric and (b) hexagonal refinement.

We examine two families of refinement. A refinement r is b -weakly bounded if $|\text{elnc}_r(e)| \leq b$ for all $e \in E$, and r is (b, c) -strongly bounded if it is b -weakly bounded and for any edge $e \in E$ and any vertex $v' \in V'$ bounding e or incident to e under r , we have that $\text{elncAdj}_r(e)$ contains at most c edges attached to v' lying in any one cell. Basically, weakly bounded means that r partitions the edges of G into at most b pieces. Strongly bounded further assumes a bound on the number of new edges attached to the boundary of each cell of G . Note that even if G does not have bounded degree, then it is possible in a strongly bounded refinement for the number of edges attached to a vertex to be unbounded; it is bounded within each cell, but there may be unboundedly many cells. Examples of strongly bounded refinements include the identity refinement, barycentric subdivision, and hexagonal refinement. See Fig. 2.

These are our notions of “sufficiently reasonable” mentioned in the introduction. Weakly and strongly bounded refinements will naturally correspond to vertex and edge extremal length, respectively. Indeed, the method He and Schramm offer in [6] to construct a VEL-parabolic, EEL-hyperbolic triangulation amounts to a weakly but not strongly bounded refinement of a VEL-parabolic graph.

3.4. Edge extremal length

Theorem 3.2. Let A be a finite combinatorial annulus and r a (b, c) -strongly bounded refinement. Then there is a constant $k \geq 1$, depending only on b and c , such that $\text{EEL}(rA)$ and $\text{EEL}(A)$ are k -comparable. That is,

$$\frac{1}{k} \text{EEL}(rA) \leq \text{EEL}(A) \leq k \text{EEL}(rA).$$

Proof. Suppose μ' is an edge extremal metric on rA . We show the function

$$\mu(e) = \max_{e' \in \text{elnc}_r(e)} \mu'(e')$$

defines an edge metric on A that provides the necessary bound. First,

$$\text{area}(\mu) = \sum_{e \in E} \mu(e)^2 = \sum_{e \in E} \max_{e' \in \text{elnc}_r(e)} \mu'(e')^2 \leq \sum_{e' \in rE} \mu'(e')^2 = \text{area}(\mu').$$

The inequality holds because no edge in rE is incident to more than one edge in E .

Now suppose $\gamma \in \Gamma = \Gamma_E(A)$. Construct the shadow path $\gamma' \in \Gamma' = \Gamma_E(rA)$ replacing each $e \in \gamma$ with the appropriately ordered edges of $\text{elnc}_r(e)$. Then

$$\begin{aligned} L_\mu(\gamma) &= \sum_{e \in \gamma} \mu(e) = \sum_{e \in \gamma} \max_{e' \in \text{elnc}_r(e)} \mu'(e') = \frac{1}{b} \sum_{e \in \gamma} b \max_{e' \in \text{elnc}_r(e)} \mu'(e') \\ &\geq \frac{1}{b} \sum_{e \in \gamma} \sum_{e' \in \text{elnc}_r(e)} \mu'(e') = \frac{1}{b} \sum_{e' \in \gamma'} \mu'(e') = \frac{1}{b} L_{\mu'}(\gamma') \end{aligned}$$

and so by Lemma 3.1

$$\text{EEL}(A) \geq \frac{1}{b^2} \text{EEL}(rA).$$

Conversely, suppose μ is an edge extremal metric on A and define an edge metric μ' on rA by

$$\mu'(e') = \begin{cases} \mu(e) & \text{if there is an } e \in E \text{ such that } e' \in \text{elnc}_r(e). \\ \max(\mu(e_1), \mu(e_2), \mu(e_3)) & \text{if } e' \subset \Delta(e_1, e_2, e_3) \text{ and is incident-adjacent to one of the } e_1, e_2, \text{ or } e_3. \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \text{area}(\mu') &= \sum_{e' \in rE} \mu'(e')^2 \leq \sum_{e \in E} \sum_{e' \in \text{elnc}_r(e)} \mu(e)^2 + \sum_{e \in E} 6c\mu(e)^2 \\ &\leq b \text{area}(\mu) + 6(1+b)c \text{area}(\mu) = (b + 6c(1+b)) \text{area}(\mu). \end{aligned}$$

The first term of the inequality comes from the fact that every edge in E contributes its measure to at most b incident edges. The second term counts the incident-adjacent edges. An edge $e \in E$ lies on the boundary of two cells τ_1, τ_2 and may contribute its measure to elements of rE that are incident-adjacent to any of the six edges bounding τ_1 and τ_2 (double counting e , once for each cell it bounds) and are contained within one of these cells. That makes six edges, each partitioned at most b times, and at most c incident-adjacent edges attached to any refined vertex lying in a given cell. Noting that b only counts the new vertices in the refinement, we also get another possible $6c$ incident-adjacent edges off of the original vertices of the τ_i , making a total contribution of $6c(1+b)$.

Now suppose $\gamma' \in \Gamma'$. Write γ' as a concatenation of segments $\gamma'_0, \sigma'_0, \gamma'_1, \sigma'_1, \dots, \gamma'_n, \sigma'_n, \dots$ such that for every $i \geq 0$ there is an $e_i \in E$ with $\sigma_i \subset \text{elnc}_r(e_i)$ and there exist $e_1^i, e_2^i \in E$ (not necessarily distinct) such that the two end edges of γ'_i are contained in $\text{elncAdj}_r(e_1^i)$ and $\text{elncAdj}_r(e_2^i)$, and all edges in γ'_i lie inside the same triangular cell. (Some of the σ_i and γ_i may be empty.) Define γ_i to be the edge path $\{e_1^i, e_2^i\}$ and $\sigma_i = \{e_i\}$. Let γ be the concatenation $\gamma_0 \sigma_0 \gamma_1 \sigma_1 \dots$. Then

$$\begin{aligned} L_{\mu'}(\gamma') &= \sum_{i \geq 0} L_{\mu'}(\gamma'_i) + \sum_{i \geq 0} L_{\mu'}(\sigma'_i) = \sum_{i \geq 0} \left(\sum_{x \in \gamma'_i} \mu'(x) + \sum_{y \in \sigma'_i} \mu'(y) \right) \\ &\geq \sum_{i \geq 0} (\mu(e_1^i) + \mu(e_2^i)) + \sum_{i \geq 0} \mu(e_i) \\ &= \sum_{i \geq 0} L_{\mu}(\gamma_i) + \sum_{i \geq 0} L_{\mu}(\sigma_i) = L_{\mu}(\gamma). \end{aligned}$$

We now apply Lemma 3.1 to obtain

$$\text{EEL}(A) \geq \frac{1}{b + 6c(1+b)} \text{EEL}(rA).$$

The proof is completed by taking $k = \max(b^2, b + 6c(1+b))$. \square

We have actually proved a little more than is stated, since the constant in the first half of the proof did not depend on c . As such, the following is a corollary of the proof.

Lemma 3.3. *Let A be a finite combinatorial annulus and r a b -weakly bounded refinement. Then $\text{EEL}(rA) \leq b^2 \text{EEL}(A)$.*

The point of this and subsequent theorems is that the comparability constant does not depend at all on A . Only the process by which A is refined matters, and even that dependence is surprisingly slight. This becomes important when we apply the theorem to the type problem.

3.5. Vertex extremal length

We would like to take the same strategy for vertex extremal length as we did for edge extremal length: for two annuli related by a bounded refinement, use an extremal metric on one to construct a new metric on the other that adequately controls its extremal length. Unfortunately in the VEL case, the degree of the original graph naturally arises in the bounds. To prove a result that is not dependent on degree, we will use the following degree property shared by all planar graphs that will be sufficient to construct the required metrics.

Lemma 3.4. *The edges of a finite planar graph $G = (V, E)$ may be directed so that the outdegree of every vertex is at most three.*

See [2] for a proof.

Theorem 3.5. *Let A be a finite combinatorial annulus and r a b -weakly bounded refinement. Then there is a constant $k \geq 1$, depending only on b , such that $\text{VEL}(rA)$ and $\text{VEL}(A)$ are k -comparable. That is,*

$$\frac{1}{k} \text{VEL}(rA) \leq \text{VEL}(A) \leq k \text{VEL}(rA).$$

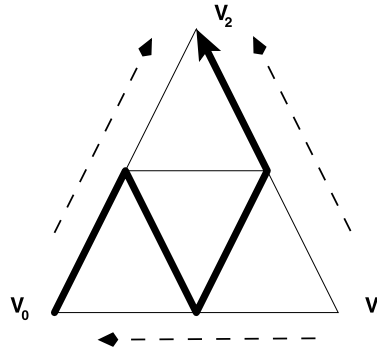


Fig. 3. An example of the shadow path in hex refinement. The bold arrow indicates five vertices in a path through a hex refined face. The dashed arrows identify how the edges are directed. The corresponding shadow path is v_0, v_0, v_1, v_1, v_2 . Both path segments have the same length in their respective metrics.

Proof. Let μ' be an extremal metric on rA . Define a vertex metric $\mu(v)$ on V to be the larger of $\mu'(v)$ and the maximal value of $\mu'(v')$ taken over all vertices $v' \in V' \setminus V$ that are incident to an edge containing v . For any $v' \in rV$, this process assigns $\mu'(v')$ to at most two vertices in V (the two vertices bounding the edge on which v' lies) and so clearly $\text{area}(\mu) \leq 2 \text{area}(\mu')$.

Let $\gamma \in \Gamma(A)$ and consider the shadow path $\gamma' \in \Gamma(rA)$ obtained by traveling between vertices of γ along the original edges of A , passing through the incident vertices. Write E_γ for the set of edges in A connecting the vertices of γ . Then

$$\begin{aligned} L_{\mu'}(\gamma') &= \sum_{v' \in \gamma'} \mu'(v') = \sum_{e \in E_\gamma} \sum_{v' \in \text{vInc}(e)} \mu'(v') + \sum_{v \in \gamma' \cap V} \mu'(v) \\ &\leq \sum_{e \in E_\gamma} b \max_{v' \in \text{vInc}(e)} \mu'(v') + \sum_{v \in \gamma' \cap V} \mu(v) \leq b \sum_{v \in \gamma} \mu(v) + \sum_{v \in \gamma} \mu(v) = (b+1)L_\mu(\gamma). \end{aligned}$$

We have thus satisfied the hypotheses of Lemma 3.1 and conclude $\text{VEL}(A) \geq \frac{1}{2(b+1)^2} \text{VEL}(rA)$.

Our examination of outdegree bounds pays off in the converse. Let μ be an extremal metric on A and direct the edges of G so that the outdegree of every vertex is at most 3. This is possible by Lemma 3.4. Define a metric μ' on rG by

$$\mu'(v) = \begin{cases} \mu(v) & \text{if } v \in V. \\ \mu(w) & \text{if } v \in rV \setminus V \text{ and } v \text{ is incident to a directed edge with tail } w. \\ 0 & \text{otherwise.} \end{cases}$$

The bounded outdegree gives the area restriction:

$$\begin{aligned} \text{area}(\mu') &= \sum_{v \in rV} \mu'(v)^2 = \sum_{v \in V} \mu'(v)^2 + \sum_{v \in rV \setminus V} \mu'(v)^2 \\ &\leq \sum_{v \in V} \mu(v)^2 + \sum_{v \in V} 3b\mu(v)^2 = (1+3b) \text{area}(\mu) = 1+3b. \end{aligned}$$

Let $\gamma' \in \Gamma(rA)$. We need to find a shadow path $\gamma \in \Gamma(A)$ with $kL_\mu(\gamma) \geq L_{\mu'}(\gamma')$ for a k depending only on b .

Construct γ inductively. Suppose γ'_i is an initial segment of γ' and assume γ_i has been constructed so that $L_{\mu'}(\gamma'_i) = L_\mu(\gamma_i)$. Let v be the endpoint of γ_i and v' the endpoint of γ'_i . If $v' \in V$, assume $v' = v$. Otherwise, $v' \in rV \setminus V$ and we assume that if v' is incident to an edge, then that edge has v as an endpoint. We want to extend γ_i by a vertex w to create a new segment γ_{i+1} so that these properties are preserved and so that $L_{\mu'}(\gamma'_i) = L_\mu(\gamma_i)$. The basic rule of the construction is “always move to the tail of the arrow.” That is, we add to γ the vertex at the tail of the directed edge every time γ' hits an incident vertex. Fig. 3 illustrates the process.

Let w' be the endpoint of γ_{i+1} . There are three cases. If w' is not incident to any edge, then $\mu'(w') = 0$ and we do nothing. Set $\gamma_{i+1} = \gamma_i$ and note $L_\mu(\gamma_{i+1}) = L_{\mu'}(\gamma'_{i+1})$ because we do not add any μ -length.

If $w' \in V$, we take $w = w'$. This is legal by the assumption that v' and v are in the same face and that G is a triangulation (guaranteeing $v \sim w$). Then $L_\mu(\gamma_{i+1}) = L_{\mu'}(\gamma'_{i+1})$ because we are adding the same measure to both.

The final case is that w' is incident to an edge e of a face containing v as vertex. In this case, we take w to be the tail of the directed edge e (γ may move or “sit and wait” at some vertex). Again, $L_\mu(\gamma_{i+1}) = L_{\mu'}(\gamma'_{i+1})$ because we are adding the same measure to both paths.

Thus we have constructed γ so that $L_\mu(\gamma) = L_{\mu'}(\gamma')$. Lemma 3.1 now applies to give $\text{VEL}(rA) \geq \frac{1}{1+3b} \text{VEL}(A)$. This proves the theorem with $k = \max(1+3b, 2(b+1)^2)$. \square

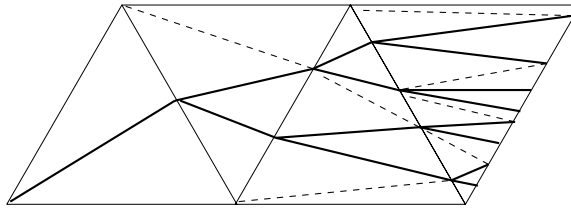


Fig. 4. Adding a binary tree with an unbounded refinement. The first four triangles of the refinement are shown. Bold edges show the binary tree. The dashed edges are added to make the refined graph a triangulation.

4. Refinement and type

4.1. Bounded refinement preserves type

Our main application is to the type problem.

Theorem 4.1. *Let G be a disk triangulation graph. If r is a weakly bounded refinement, then G and rG have the same VEL type. If r is a strongly bounded refinement, then G and rG have the same EEL type.*

Proof. We prove the VEL case. The EEL case is identical.

Let C be the vertex cycle formed from the neighbors of a base vertex v_0 . Let $\{A_i\}_{i=0}^\infty$ be a collection of combinatorial annuli, each with innermost boundary component C , such that $G \setminus \bigcup_{i \geq 0} A_i = \{v_0\}$. Apply Theorem 3.5 (Theorem 3.2 for EEL) to conclude that $\text{VEL}(A_i)$ and $\text{VEL}(rA_i)$ are k -comparable for some $k > 0$ depending only on r . The collections $\{A_i\}_{i=0}^\infty$ and $\{rA_i\}_{i=0}^\infty$ then satisfy the hypotheses of Lemma 2.3, which says $\bigcup_{i \geq 0} A_i$ and $\bigcup_{i \geq 0} rA_i$ have the same type. The theorem follows. \square

When discussing a disk triangulation graph G , we will denote by $\Gamma_\infty(G)$ the collection of (edge or vertex) paths from a chosen base point to infinity (i.e., that eventually leave every finite subgraph of G).

4.2. Unbounded refinements

We now present some examples illustrating the necessity of bounded refinements in preserving type.

Theorem 4.2. *Every disk triangulation graph G has a hyperbolic refinement ζG .*

Proof. Let G be a VEL- or EEL-parabolic disk triangulation graph and let T_1, T_2, \dots be an infinite collection of distinct faces such that for each $i > 0$, T_i intersects T_{i+1} along a single edge e_{i+1} , and so that T_i and T_j share an edge only if $|i - j| = 1$. Let v_0 be the vertex of T_1 that is not in e_1 .

Refine the T_i by attaching 2^i vertices to each e_i and connecting each new vertex to exactly two of the new vertices attached to e_{i+1} via an edge contained in T_{i+1} . See Fig. 4. The new vertices and edges form a binary tree, which is VEL- and EEL-hyperbolic. Since the refined graph contains a hyperbolic graph, it must be hyperbolic by the monotonicity property. The refined graph may be made into a disk triangulation graph by adding more edges in the T_i to divide the quadrilaterals formed between the branches of the tree into triangles. \square

Theorem 4.3. *For any infinite graph G it is possible to attach vertices to the edges of G to form a graph that is VEL-parabolic.*

Proof. Let G be an infinite graph and let $\{A_i\}_{i \geq 0}$ be a collection of disjoint finite sets of edges with the property that every path $\gamma \in \Gamma_\infty(G)$ intersects each of the A_i along at least one edge. For example, we may take A_i to be the set of edges with one endpoint in the sphere of radius 2^i and the other in the sphere of radius 2^{i+1} . Let k_i be the number of edges in A_i and consider the graph ζG that adds k_i vertices incident to each edge in A_i for every $i \geq 0$, and leaves any edges not in any A_i untouched. The trick is to let the size of the A_i 's prescribe exactly how much to slow the vertex growth.

Define a metric μ on ζG by

$$\mu(v) = \begin{cases} \frac{1}{ik_i} & \text{if } v \text{ is incident to an edge in } A_i, \\ 0 & \text{otherwise.} \end{cases}$$

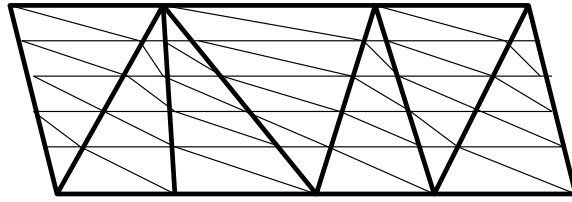


Fig. 5. The zig-zag refinement ζ_4 . The horizontal lines are the levels. The diagonal lines ensure that the refinement is a triangulation.

We show ζG is parabolic by proving that μ is an extremal metric. Let $\gamma' \in \Gamma_\infty(\zeta G)$. By construction, for every positive integer i there are adjacent vertices $v_i, w_i \in \gamma' \cap A_i$. Thus for each $i \geq 0$, γ' intersects at least k_i vertices incident to an edge in A_i . Hence,

$$L_\mu(\gamma') = \sum_{v \in \gamma'} \mu(v) \geq \sum_{i=0}^{\infty} k_i \cdot \frac{1}{ik_i} = \sum_{i=0}^{\infty} \frac{1}{i} = \infty.$$

On the other hand,

$$\begin{aligned} \text{area}(\mu) &= \sum_{v \in V(\zeta G)} \mu(v)^2 = \sum_{i=0}^{\infty} \sum_{v \in A_i} \mu(v)^2 \\ &= \sum_{i=0}^{\infty} k_i \cdot k_i \cdot \left(\frac{1}{ik_i}\right)^2 = \sum_{i=0}^{\infty} \frac{1}{i^2} < \infty. \end{aligned}$$

The two k_i 's beginning the second line reflect each of the k_i edges of A_i being refined k_i times. This makes μ a finite area metric such that all paths to infinity have infinite length, hence ζG has a parabolic extremal metric. \square

The construction described adds only vertices. We now sketch how to mimic this effect on a refined triangulation.

Theorem 4.4. *Let G be a VEL-hyperbolic disk triangulation graph. Then there is a refinement of G that yields a VEL-parabolic disk triangulation graph.*

Proof. Let A_i be the set of edges with one endpoint in the sphere of radius 2^i and the other in the sphere of radius 2^{i+1} . Consider the “zig-zag” refinement ζ_n depicted in Fig. 5. This refinement forms a triangulation in which n “levels” are added between two cycles. Let ζG be the graph formed by applying the ζ_{k_i} refinement to the A_i , where k_i is the number of edges in A_i , and leaving untouched any edge not in some A_i . The proof now proceeds exactly as in Theorem 4.3. Details are left to the reader, or see [14]. \square

5. Extensions and applications

5.1. General cell complexes

The proof of Theorem 3.5 depends in an essential way on the assumption that G was a triangulation. The significant feature of triangulations is that when a refined path cuts through a face near some vertex v , it must leave again near a vertex adjacent to v . This is no longer the case if we allow complexes with non-triangular faces, in which refined paths may cut through the face and emerge near vertices too far away from its entry point to be counted properly. In other words, our directed graph trick no longer works.

We can still say something if we permit some assumptions on the planar complex G . For a cell complex $G = (V, E, F)$, we define its *dual complex* $G^* = (F, E^*, V)$ to be the complex whose vertices are the faces of G and whose edges are the pairs of faces of G sharing an edge of G . We say G has (d, a) -*dually bounded degree* if G has degree d and the dual complex G^* has degree a . The latter condition is equivalent to requiring that the faces of G each have at most a sides. For example, a bounded degree triangulation is $(d, 3)$ -dually bounded for some d . A graph is *dually bounded* if it is (d, a) -dually bounded for some d, a . Our definitions for bounded refinements of cell complexes are just as for refinements of triangulations defined in Section 3.3.

We now offer results for dually bounded complexes similar to those we have already obtained for triangulations. The arguments are also similar and we leave the reader to fill out the proof sketches below or see [14].

Theorem 5.1. *Let $A = (V, E, F)$ be a (d, a) -dually bounded finite annular complex and r a b -weakly bounded refinement of A . Then $\text{VEL}(rA)$ and $\text{VEL}(A)$ are k -comparable for some k depending only on a, b , and d .*

Proof. We assume for convenience that $b > 0$ (if $b = 0$, we may take $b = 1$).

Let μ' be an extremal vertex metric on $rA = (V', E', F')$. Define $\mu(v)$ to be the greater of $\mu'(v)$ and the maximum value of $\mu'(v')$ taken over all $v' \in V'$ incident to some edge $[v, w]$, $w \in V$. An argument similar to that of Theorem 3.5 gives

$$\text{VEL}(rA) \leq \frac{(b+1)^2}{2} \text{VEL}(A).$$

Conversely, let μ be an extremal vertex metric on $A = (V, E, F)$. Let $rA = (V', E')$ and for each $v' \in V'$ define the *face neighbors* of v' as the set $f(v') = \{w' \in V' : w', v' \text{ lie incident to or within the boundary of some face of } A\}$. Define a vertex metric μ' on rA by

$$\mu'(v') = \begin{cases} \max_{w' \in f(v')} (\mu(w)) & \text{if } v' \in V \text{ or } v' \text{ is incident to an edge of } E, \\ 0 & \text{otherwise.} \end{cases}$$

Again, the argument in the proof of Theorem 3.5 gives

$$\text{VEL}(rA) \geq \frac{1}{a^3 b d} \text{VEL}(A),$$

proving the theorem with $k = \max(\frac{1}{2}(b+1)^2, a^3 b d)$. \square

Degree is not a problem for edge extremal length, but we still require a strongly bounded refinement and a bounded degree dual.

Theorem 5.2. *Let A be a finite annular complex and r a (b, c) -strongly bounded refinement of G . Suppose every face of G has at most a sides. Then $\text{EEL}(rG)$ and $\text{EEL}(A)$ are k -comparable for some k depending only on a, b , and c .*

Proof. The proof of the relation

$$\text{EEL}(rA) \leq b^2 \text{EEL}(A)$$

in Theorem 3.2 did not depend on the cells being triangular. We take this as proved.

For the reverse relation, let μ be an edge extremal metric on A . For $e' \in E'$, define $\mu'(e')$ as

$$\mu'(e') = \begin{cases} \mu(e) & \text{if there is an } e \in E \text{ such that } e' \in \text{elnc}_r(e), \\ \max(\mu(e_1), \dots, \mu(e_n)) & \text{if } e' \text{ lies in the interior of a face bounded by edges } e_1, \dots, e_n \text{ and} \\ 0 & \text{if } e' \text{ is incident-adjacent to one of the } e_i. \\ & \text{otherwise.} \end{cases}$$

We leave it to the reader to adapt the proof of Theorem 3.2 to obtain the relation

$$\text{EEL}(A) \leq \frac{1}{4} a^2 (2ac(b+1) + b) \text{EEL}(rA).$$

The theorem thus holds for $k = \max(b^2, \frac{1}{4} a^2 (2ac(b+1) + b))$. \square

By filling out a graph with annuli as before, we get the corresponding statements on type.

Corollary 5.3. *Let G be an infinite planar complex.*

1. *If r is a weakly bounded refinement of G and G is dually bounded, then G and rG have the same VEL type.*
2. *If r is a strongly bounded refinement of G and G^* has bounded degree, then G and rG have the same EEL type.*

Note that if a graph is dually bounded then its VEL and EEL types are the same and we may meaningfully refer to the combinatorial type of the graph.

5.2. Dual graphs

Corollary 5.3 may be applied to relate the combinatorial type of a complex to that of its dual.

Theorem 5.4. *A dually bounded planar cell complex has the same combinatorial type as its dual complex G^* .*

Proof. Let $G = (V, E, F)$ be a dually bounded planar cell complex. Consider the refinement G° constructed by adding a vertex v_f inside each face of $f \in F$ and a vertex v_e incident to each edge $e \in E$. Connect these new vertices by adding edges of the form $[v_f, v_e]$, where e is an edge bounding the face f . Roughly, we are superimposing G^* onto G and attaching

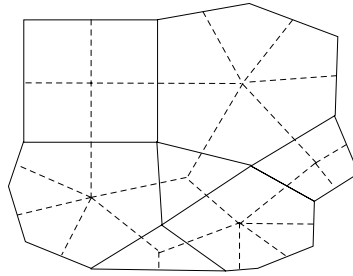


Fig. 6. The graph G^\diamond is a refinement of both G and G^* . Solid lines indicate edges of a complex G , dashed lines indicate the dual complex G^* .

vertices where they intersect. See Fig. 6. This construction clearly defines a strongly bounded refinement $rG = G^\diamond$ of G , so G^\diamond and G have the same type by Corollary 5.3. We associate v_f to its dual vertex $f \in V^* = F$, a pair of edges $[v_f, v_e], [v_g, v_e]$, $e \in E$, $f, g \in F$ to the dual edge $[f, g] \in E^*$, and we note that each vertex $v \in V$ lies inside a distinct face formed by the edge pairs associated to E^* . With these associations, we see that G^\diamond is isomorphic not only to rG , but also to $r(G^*)$ – the vertices v_e attached to an edge e are identified with vertices v_{e^*} attached to the dual edges, and we similarly reverse the roles of vertices and faces. We have $G^\diamond = rG \cong r(G^*)$, and so G and G^* have the same type again by Corollary 5.3. \square

5.3. Outer spheres

The refinement theorems we have developed suggest solving the combinatorial type problem for a specific graph G by finding a dually bounded subcomplex of G from which G can be obtained by a bounded refinement. The purpose of this section is to construct a candidate subgraph.

Let $G = (V, E)$ be a disk triangulation graph with distinguished base vertex v_0 . Define the *outer sphere of radius n* $S_O(n)$ to be the collection of vertices $v \in G$ such that $|v| = n$ and for which there is a path $\gamma^+(v)$ from v to infinity containing no other vertices of norm n . The edges of $S_O(n)$ are the edges of E whose vertices both lie in $S_O(n)$.

The typical spheres $S(n) = \{v : |v| = n\}$ may be massively disconnected because of geodesics that cannot be extended, mimicking a classical phenomenon. The outer spheres ignore these unwanted components. We show that the outer spheres are cycle graphs.

Lemma 5.5. $S_O(n)$ is a cycle graph for all integers $n > 0$.

Proof. Fix n and let $v \in S_O(n)$. Note that v is the only vertex in $\gamma^+(v)$ whose norm is not strictly larger than n . To see this, observe that since $\gamma^+(v)$ goes to infinity, it must contain vertices of arbitrarily large norm. Were there a vertex of norm less than n , then the fact that adjacent vertices differ in norm by at most 1 implies that there would also be a vertex of norm n , contradicting the definition of $\gamma^+(v)$.

Similarly, for every vertex $v \in S_O(n)$ there is a path $\gamma^-(v)$ connecting v_0 to v such that $\gamma^-(v)$ is of length n and therefore contains no other vertices with norm greater than or equal to n . This is immediate from the definitions of $S_O(n)$ and norm. Altogether, the concatenated path $\gamma(v) = \gamma^-(v) \cup \gamma^+(v)$ connects the base point v_0 to infinity so that v is the only vertex in the path of norm n , all vertices before v in $\gamma(v)$ have norm less than n , and all vertices after v have norm larger than n .

The successor w and predecessor u of v in $\gamma(v)$ must therefore have norms $n+1$ and $n-1$, respectively. Since G is a triangulation, the set of neighbors of v are cyclicly connected by edges and so the closed loop connecting the neighbors of v contains at least one vertex each of norm $n+1$ and $n-1$. These vertices divide the loop of neighbors of v into two segments, each containing a vertex with norm n , again because successive vertices along a path may differ in norm by at most 1. So v has at least two neighbors v_1, v_2 with norm n . Start at v_1 and proceed around the loop of edges toward w . We may assume v_1 was chosen so that no other vertices of norm n are encountered. Then travel along $\gamma^+(v)$ away from v , giving a path from v_1 to infinity that contains no vertices with norm n . Repeating for v_2 , we have shown that any v in $S_O(n)$ has at least two neighbors in $S_O(n)$, i.e. that every vertex in $S_O(n)$ has degree at least two. See Fig. 7.

Now suppose for contradiction that $v \in S_O(n)$ has three neighbors w_1, w_2, w_3 in $S_O(n)$ connected to v by edges e_1, e_2, e_3 . Then to each w_i there is a path γ_i^+ connecting w_i to infinity whose interior vertices all have norm at least n . Assume without loss of generality that the γ_i^+ do not contain the base point v_0 . The set $\mathbb{C} \setminus (\gamma_1^+ \cup \gamma_2^+ \cup \gamma_3^+ \cup \{e_1, e_2, e_3\})$ is a collection of regions in the plane whose boundaries contain only vertices of norm greater than or equal to n and at most two of the w_i . See Fig. 8.

Consider the region R containing the base point v_0 and suppose w_1 is not in its boundary ∂R . Then by assumption there is a path $\gamma^-(w_1)$ connecting v_0 to w_1 and containing only vertices with norm less than n except the endpoints. But ∂R separates v_0 from w_1 , so $\gamma^-(w_1)$ must intersect ∂R , all of whose vertices have norm greater than or equal to n . This is a contradiction, so v cannot have degree greater than two. Since $S_O(n)$ is compact (because G is locally finite) and all

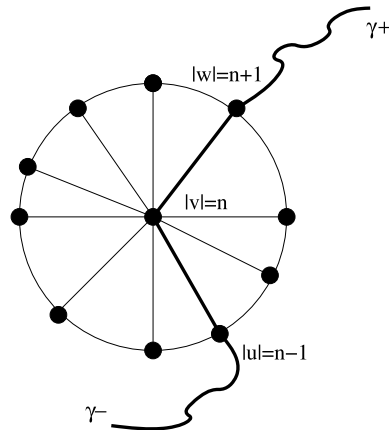


Fig. 7. A vertex in $S_{\mathcal{O}}(n)$ has two neighbors in $S_{\mathcal{O}}(n)$.

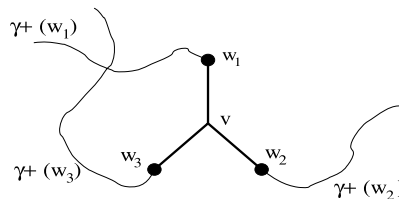


Fig. 8. A vertex in $S_{\mathcal{O}}(n)$ cannot have three neighbors in $S_{\mathcal{O}}(n)$.

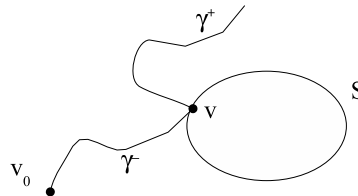


Fig. 9. Both paths lie in the unbounded component of $\mathbb{C} \setminus S$.

of its vertices have degree two, $S_{\mathcal{O}}(n)$ must be a union of disjoint cycle graphs. We have only to show $S_{\mathcal{O}}(n)$ has but one component.

We begin by showing that if S is a component of $S_{\mathcal{O}}(n)$, then v_0 is contained in the bounded component of $\mathbb{C} \setminus S$. Suppose for contradiction that v_0 is in the unbounded component and let $v \in S$. As before, there is a path $\gamma^-(v)$ from v_0 to v with all interior vertices having norm smaller than n , and a path $\gamma^+(v)$ from v_0 to infinity with all interior vertices of norm larger than n . Then the paths $\gamma^+(v)$ and $\gamma^-(v)$ both lie entirely in the unbounded component except where they meet at v . But then the successor of v in $\gamma^+(v)$ and the predecessor of v in $\gamma^-(v)$ both lie in the unbounded component of $\mathbb{C} \setminus S$. These vertices have norm $n+1$ and $n-1$, respectively, and by the argument used above, we have that there must be another element of $S_{\mathcal{O}}(n)$ lying in the unbounded component of S , a contradiction to the assumption that S is a cycle. See Fig. 9.

The only remaining possibility is that the components of $S_{\mathcal{O}}(n)$ are concentric. But if there is more than one component, then it is not possible to find a path to infinity from a vertex in an inner component without crossing the outermost component which contains only vertices of norm n . We are left to conclude that $S_{\mathcal{O}}(n)$ is connected, proving the claim. \square

For a disk triangulation graph G , its outer spheres suggest a subgraph of G for study. Define the *outer sphere skeleton* $G_{\mathcal{O}}$ to be the union of the outer spheres $S_{\mathcal{O}}(n)$ along with all edges of the form $[v_n, v_{n+1}]$ where $v_n \in S_{\mathcal{O}}(n)$ and $v_{n+1} \in S_{\mathcal{O}}(n+1)$.

$G_{\mathcal{O}}$ is simply the set of outer spheres along with the edge geodesics connecting the outer spheres to v_0 . It discards the isolated face subdivisions that we have already seen cannot impact VEL type for dually bounded degree complexes. $G_{\mathcal{O}}$ has an appealing structure. All vertices on $S_{\mathcal{O}}(n)$ may be traced back to the base vertex v_0 by working backward through each of the previous outer spheres. Each face of $G_{\mathcal{O}}$ lies between two outer spheres $S_{\mathcal{O}}(n)$ and $S_{\mathcal{O}}(n+1)$. The face is bounded by two edges connecting these spheres, at most one edge of $S_{\mathcal{O}}(n+1)$, and any number of consecutive edges along $S_{\mathcal{O}}(n)$. The following is a special case of Theorem 5.3.

Theorem 5.6. Let G be a disk triangulation graph. If the outer sphere skeleton $G_{\mathcal{O}}$ is VEL-hyperbolic, then G is VEL-hyperbolic. If $G_{\mathcal{O}}$ is dually bounded and VEL-parabolic, then G is VEL-parabolic.

6. Growth and type

We now consider to what extent combinatorial type can be determined from the growth of a graph. There are several results that say roughly that slow-growing graphs are parabolic and fast-growing graphs are hyperbolic. For example, Rodin and Sullivan [7] showed using circle packings (which connect to vertex extremal length by [6]) that linear growth of spheres in a bounded degree graph implies parabolicity, whereas Siders [10] used electrical methods (via [5]) to determine the type of a graph formed by interspersing cycles of 6- and 7-degree vertices. See also [13]. In this section, we establish a sharp parabolicity condition that requires no symmetry or degree restrictions of the graph.

Define $\log_{(0)}(x) = x$ and $\log_{(m+1)}(x) = \log(\log_{(m)}(x))$ for any positive integer m . Inductive application of the chain rule gives the derivative $\log'_{(m+1)}(x) = \frac{d}{dx} \log_{(m+1)}(x) = (\log_{(0)}(x) \log_{(1)}(x) \cdots \log_{(m)}(x))^{-1}$, and define the *generalized p -series*

$$\wp(m, p) = \sum_{j=e_m}^{\infty} \frac{\log'_{(m)}(j)}{(\log_{(m)}(j))^p} = \sum_{j=e_m}^{\infty} \frac{1}{j(\log j)(\log \log j) \cdots (\log_{(m-1)} j)(\log_{(m)} j)^p}$$

where e_m is the smallest integer for which $\log_{(m)}(e_m)$ is defined.

Lemma 6.1. Let m be a nonnegative integer and $c > 0$ such that $\log_{(m)}(c) > 0$.

1. The integral

$$\int_c^{\infty} \frac{\log'_{(m)}(x)}{(\log_{(m)}(x))^p} dx$$

converges if and only if $p > 1$.

2. The generalized p -series $\wp(m, p)$ converges if and only if $p > 1$.

Proof. Making the substitution $u = \log_{(m)}(x)$ and $du = \log'_{(m)}(x) dx$, we integrate

$$\int_c^{\infty} \frac{\log'_{(m)}(x)}{(\log_{(m)}(x))^p} dx = \int_{\log_{(m)}(c)}^{\infty} \frac{1}{u^p} du,$$

which converges if and only if $p > 1$. This proves the first part of the lemma, and the second part follows because the generalized p -series approximates this integral. See [8] for an alternate proof and general discussion. \square

We can now establish the parabolicity condition.

Theorem 6.2. Let $G = (V, E)$ be a graph with distinguished vertex v_0 . Suppose there is a collection of subgraphs $\{C_j = (V_j, E_j)\}$ of G such that for each $j \in \mathbb{N}$, every vertex path from v_0 to infinity intersects C_j at least once. If there is a positive integer $m \in \mathbb{N}$ and a positive constant $K > 0$ such that for every $j \geq e_m$ we have $|V_j| \leq Kj(\log j)(\log \log j) \cdots (\log_{(m-1)} j) = \frac{K}{\log_{(m)} j}$, then G is VEL-parabolic.

Proof. Define a vertex metric μ on G by

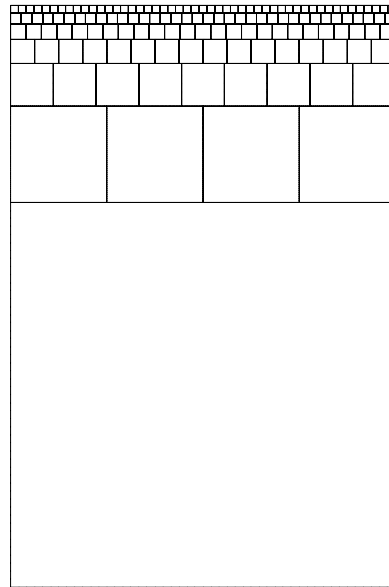
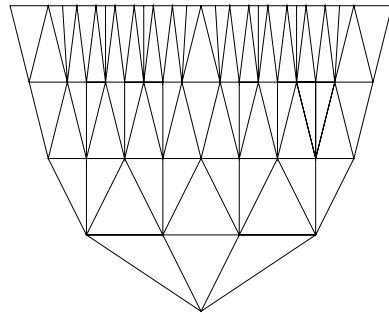
$$\mu(v) = \begin{cases} \frac{\log'_{(m)}(j)}{\log_{(m)}(j)} & \text{if } v \in V_j \text{ and } j \geq e_m, \\ 0 & \text{otherwise.} \end{cases}$$

Let Γ be the collection of transient vertex paths in G based at v_0 . Since each $\gamma \in \Gamma$ must contain at least one vertex in each of the V_j , we have

$$L_{\mu}(\gamma) = \sum_{v \in \gamma} \mu(v) \geq \sum_{j=e_m}^{\infty} \frac{\log'_{(m)}(j)}{\log_{(m)}(j)} = \wp(m, 1).$$

This sum diverges by Lemma 6.1, showing all paths in Γ have infinite length.

We now show that μ has finite area.

(a) A square tiling with n^2 squares in the n th row

(b) The corresponding triangulation

Fig. 10. A hyperbolic triangulation with quadratic growth.

$$\text{area}(\mu) = \sum_{v \in V} \mu(v)^2 = \sum_{j=e_m}^{\infty} \frac{|V_j|(\log'_{(m)}(j))^2}{(\log_{(m)}(j))^2} \leq \sum_{j=e_m}^{\infty} \frac{K \log'_{(m)}(j)}{\log_{(m)}(j)^2} = K \wp(m, 2)$$

which is finite by Lemma 6.1.

It follows that μ is a parabolic extremal vertex metric for G and so G is VEL-parabolic. \square

Of particular note in Theorem 6.2 is the case where G is a disk triangulation graph and C_j is the outer sphere of radius j about v_0 . Note also that the theorem does not hold in the EEL context as stated, but the proof can be adapted easily if we require slow growth of the number of edges emanating from the cycles C_j . Details are an exercise.

There is no hope for a converse to Theorem 6.2, as P. Soardi [11] has a parabolic graph with exponential growth. The example has bounded degree and thus functions in both the VEL and EEL settings.

We can get a sharpness result, however.

Theorem 6.3. *Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of positive integers such that $\sum_{i=1}^{\infty} \frac{1}{a_i} < \infty$. Then there is a VEL-hyperbolic disk triangulation graph with a vertex $v_0 \in G$ such that the n -sphere about v_0 is a cycle containing at most a_n vertices.*

Proof. Assume for convenience that $\{a_i\}$ is monotone. The construction uses a connection of vertex extremal length to square tilings established in [3] and [9]: if $G = (V, E)$ is a triangulation of a finite closed quadrilateral, then there is a tiling by squares of a rectangle so that each vertex in V corresponds to a square in the tiling and two squares intersect if and only if their corresponding vertices determine an edge in E . The side lengths of the squares determine an extremal metric for G , and the vertex extremal length of G is the aspect ratio of the rectangle.

Construct a tiling $T_n, n > 1$ as follows. Begin with a unit square, which corresponds to v_0 . Add a row of a_1 squares with side lengths all $\frac{1}{a_1}$ along one side of the unit square. Continuing adding rows of a_i squares with side lengths $\frac{1}{a_i}$ for all $i \leq n$.

Fig. 10 illustrates the construction for $a_n = (n+1)^2$. Minor adjustments to some of the a_i may be necessary in the unlucky case that some a_i divides a_{i+1} and the corresponding graph is not a triangulation.

The sides of the tiled rectangle are 1 and $1 + \sum_{i=1}^n \frac{1}{a_i}$, so the vertex extremal length of the corresponding triangulation is $1 + \sum_{i=1}^{\infty} \frac{1}{a_i}$. By assumption, this remains finite as $n \rightarrow \infty$ and so the limiting triangulation T_{∞} is VEL-hyperbolic. This triangulation forms the required disk triangulation graph if we identify the pairs of boundary vertices that lie in common spheres about v_0 . \square

We have thus shown, for example, that $O(n)$ sphere growth implies a graph is parabolic, whereas $O(n^{1+\varepsilon})$ sphere growth is indeterminate for $\varepsilon > 0$.

Note that the construction also works in the EEL setting because VEL-hyperbolic implies EEL-hyperbolic.

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